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Receding Horizon Guidance Laws with No Information on the Time-to-Go

Ki Baek Kim, Myung-Joon Kim, and Wook Hyun Kwon
Seoul National University, Seoul 151-742, Republic of Korea

In this paper a modified optimal guidance law is proposed that does not use information on the time-to-go. The proposed receding horizon guidance law (RHG) is based on the receding horizon strategy and optimal control theory. In the presence of arbitrary target maneuvers and an initial lateral miss distance (MD) rate between the target and the missile, the proposed RHG is shown to guarantee to keep the lateral MD less than the given value, within which the warhead of the missile is detonated, from the appropriately selected time to the intercept time. Through three simulation examples the ability of the RHG to intercept the target is illustrated. The performance of the RHG is compared with the optimal guidance law in terms of the terminal MD when the time-to-go is inaccurate, and in terms of the lateral MD and the missile acceleration when the time-to-go is accurate.

I. Introduction

FOR practical applications of a missile guidance law, the proportional navigation guidance law (PNG) has been widely used for more than a few decades because its implementation is simple and its cost is low. The PNG is known to be an optimal solution, which, in the absence of autopilot lags and target maneuvers, minimizes the linear quadratic cost function of the missile acceleration.[1] However, the PNG has a problem in that it cannot use information on future target maneuvers or on autopilot lags even if they are known. Hence, the PNG requires large accelerations to intercept a target that maneuvers arbitrarily[2] and may fail to intercept the target when there are autopilot lags.[3]

To cope with the problem of the PNG, optimal guidance laws (OGL) based on optimal control theory have been investigated widely since the 1960s, despite the fact that the OGL requires more measurement information than the PNG.[3–9] The performance of the OGL is dependent on the estimation of the time-to-go, which is commonly approximated as the range between the target and missile divided by the closing velocity. Typically, the estimates of the range and the closing velocity are obtained from radar or other ranging devices. In reality, however, the data are contaminated by noise from radar-jamming devices or in the processing electronics. This affects accuracy in the estimation of the time-to-go, which then causes errors in the terminal MD. For this reason, there has been some

Ki Baek Kim has been a researcher of the Engineering Research Center in Seoul National University since 1995. He received his B.S. and M.S. degrees in control and instrumentation from the same university in 1993 and 1995, respectively. He is currently a Ph.D. candidate in the School of Electrical Engineering of the same university. His research interests include predictive control, guidance law, application to autopilot, disturbance rejection, delayed control, and computer aided control system design.

Myung-Joon Kim has been a researcher of Samsung Electronics Company since 1998. He received his B.S., M.S., and Ph.D. degrees in control and instrumentation engineering from Seoul National University, Seoul, Korea, in 1991, 1993, and 1998, respectively. His research interests include predictive control, guidance law, autopilot, filtering, and signal processing.

Wook Hyun Kwon has been a Professor at Seoul National University since 1977. He received his B.S. and M.S. degrees in electrical engineering from Seoul National University, Seoul, Korea, in 1966 and 1972, respectively. He received the Ph.D. degree from Brown University, Providence, RI, in 1975. From 1975 to 1976, he was a Research Associate at Brown University. From 1976 to 1977, he was an adjunct assistant professor at University of Iowa, Iowa City, IA. Since 1977, he has been with the School of Electrical Engineering, Seoul National University. From 1981 to 1982, he was a visiting assistant professor at Stanford University, Stanford, CA. Since 1991, he has been the Director of the Engineering Research Center for Advanced Control and Instrumentation.

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research on the exact estimation of the time-to-go where additional measurement information and somewhat complicated algorithms are required.[10–12] These approaches, however, do not work well when measurement information is inaccurate or unavailable. Therefore, a new guidance law is required that does not use the time-to-go but is still based on optimal control theory. To the author's knowledge, this type of guidance law has not previously been investigated.

In this paper a modified optimal guidance law is proposed, which does not use the time-to-go but can handle autopilot lags and bounded arbitrary future target maneuvers. The proposed guidance law uses the receding horizon strategy,[13] which will be called a receding horizon guidance law (RHG) in this paper. The proposed RHG can intercept the target, keeping the lateral MD less than the given value, within which the warhead of the missile is detonated, from the appropriately selected time to the intercept time, even in the presence of arbitrary target maneuvers and an initial lateral MD rate. For convenience of analysis the assumption is made that the autopilot has no lags because, although both RHG and OGL can be designed to accommodate autopilot lags, it makes them very hard to analyze in terms of the lateral MD. Through simulation, the performance of the RHG is shown to be better than that of the OGL when the time-to-go is inaccurate or unavailable.

In Sec. II, the RHG, which is based on optimal control theory but does not use the time-to-go, is proposed. By using the navigation constant and the cost horizon size, a missile with the RHG can intercept the target in the presence of arbitrary target maneuvers and an initial lateral MD rate. In Sec. III, through simulation, the ability of the RHG to intercept the target is illustrated. The performance of the RHG is compared with that of the OGL in terms of MD and the missile acceleration. Finally, conclusions are presented in Sec. IV.

II. RHG

The missile-target intercept model can be depicted as in Fig. 1, where the following variables are defined: σ is the line-of-sight angle (LOS); r is the missile-to-target range (also known as the miss distance); y is the lateral MD; A_t is the target acceleration; V_t is the target velocity; A_{tn} is the target normal acceleration; V_{tn} is the target normal velocity; A_m is the missile acceleration; V_m is the missile velocity; A_{mn} is the missile normal acceleration; V_{mn} is the missile normal velocity; γ_m is the missile flight-path angle; and γ_t is the target flight-path angle.

The missile-target intercept model in Fig. 1 is described by the following nonlinear differential equations[2]:

$$\dot{r} = V_m[\rho\cos(\theta_t) - \cos(\theta_m)], \qquad r\dot{\sigma} = V_m[\rho\sin(\theta_t) - \sin(\theta_m)] \tag{1}$$

where θ_t, θ_m, and ρ are defined by

$$\theta_t = \gamma_t - \sigma, \qquad \theta_m = \gamma_m - \sigma, \qquad \rho = V_t / V_m$$

Here, θ_m is the angle between missile velocity vector and LOS, θ_t is the angle between target velocity vector and LOS, and ρ is the target-to-missile velocity ratio.

To proceed with the derivation, as shown in Refs. 3–6, and 8, the assumption is made that LOS angle $\sigma \approx 0$, perturbation angles $\Delta\gamma_m \approx 0$, $\Delta\gamma_t \approx 0$, velocities V_t and V_m are constant, and the autopilot has no lags. In this case, because $\dot{y} = V_t \sin\gamma_t - V_m \sin\gamma_m$, $A_{\text{mn}} = V_m\dot{\gamma}_m\cos\gamma_{m_0}$, and $A_{\text{tn}} = V_t\dot{\gamma}_t\cos\gamma_0$ (Ref. 11), the dynamics are expressed in terms of state variables normal to the reference:

$$\dot{X} = FX + BA_{\text{mn}} + GA_{\text{tn}} \tag{2}$$

where

$$X = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The existing optimal guidance problem subject to the dynamics (2) is to minimize

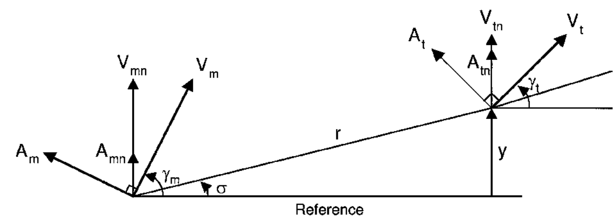

Fig. 1 Intercept geometry.

$$J = \int_0^{t_f} A_{\text{mn}}^2(t)\,\mathrm{d}t \tag{3}$$

where

$$y(t_f) = 0$$

Here, t_f is the final time to intercept the target. When the assumption is made that the target is nonmaneuvering (i.e., $A_{\text{tn}} = 0$), the OGL to the preceding problem is well known and is given as

$$A_{\text{mn}}(t) = (N/t_{\text{go}}^2)[y(t) + t_{\text{go}}\dot{y}(t)] \tag{4}$$

where $t_{\text{go}} = t_f - t$. Here, N is the navigation constant and t_{go} is the time-to-go. As shown in Eq. (4), an estimate of the final time t_f, i.e., the time-to-go t_{go}, is needed to calculate the gain of the OGL.

On the other hand, consider another guidance problem to minimize

$$J = \int_t^{t+T} A_{\text{mn}}^2(\tau)\,\mathrm{d}\tau \tag{5}$$

where

$$y(t+T) = 0$$

Here, T is an adjustable design parameter that is irrespective of the final time t_f. We then propose the RHG, which is obtained from the following procedure:

1) At the present time t the solutions for the optimal closed-loop control $A_{\text{mn}}(\tau)$ are obtained for $\forall\tau \in [t, t+T)$, which minimizes Eq. (5).

2) Among these controls only the first control $A_{\text{mn}}(\tau)_{\tau=t}$ is used.

3) At the next t the procedures 1) and 2) are repeated.

According to the preceding procedure, the proposed RHG is given as

$$\begin{aligned} A_{\text{mn}}(t) &= (N/T^2)[y(t) + T\dot{y}(t)] \\ &= [N/T^2 \quad N/T]X(t) \end{aligned} \tag{6}$$

The preceding procedure is well known as the receding horizon control scheme in control area. The receding horizon control has received much attention in both academa[13,14] and industry fields[15] because it has many advantages such as simple computation, good tracking performance, I/O constraint handling, and extension to nonlinear systems, compared with the steady-state linear quadratic control.

The following theorem shows how the proposed RHG can intercept the target even in the presence of arbitrary target maneuvers and an initial lateral MD rate.

Theorem 1: Assume that the reference is aligned with the LOS at initial time, i.e., $y(0) = 0$, that the target is arbitrarily maneuvering in the range of a positive constant α, i.e., $|A_{\text{tn}}(t)| \le \alpha$, and that the initial lateral MD rate is in the range of a positive constant β, i.e., $|\dot{y}(0)| \le \beta$.

For any $t_D > 0$ and $R_D > 0$, the proposed RHG makes the lateral MD $y(t)$ satisfy the following relation:

$$|y(t)| \le R_D \qquad \text{for} \qquad t \ge t_D \tag{7}$$

if the following equation depending on N is satisfied:

$$R_D \ge \begin{cases} \left\{ \dfrac{T}{\sqrt{N(N-4)}} \exp\left[\dfrac{-N + \sqrt{N(N-4)}}{2T}\right] t_D\beta + \dfrac{T^2}{N}\alpha \right\} & \text{for} \quad N > 4 \\ \left[\left\{ t_D e^{-(2/T)t_D}\beta \text{ for } t_D \ge \dfrac{T}{2}, \ \dfrac{T}{2}e^{-1}\beta \text{ for } t_D < \dfrac{T}{2} \right\} + \dfrac{T^2}{4}\alpha \right] & \text{for} \quad N = 4 \\ \left[\dfrac{2T}{\sqrt{N(4-N)}} e^{-(N/2T)t_D}\beta + \dfrac{4T^2}{N\sqrt{N(4-N)}}\alpha \right] & \text{for} \quad N < 4 \end{cases} \tag{8}$$

where there always exist two free parameters N and T satisfying these conditions.

The proof of Theorem 1 is given in Appendix A. Here, R_D is a lateral MD within which the missile can intercept the target. As shown in Theorem 1, the RHG guarantees $|y(t)| \leq R_D$ for $t \geq t_D$, whereas the OGL guarantees $|y(t_f)| = 0$. t_D is not the final time t_f. To intercept the target, the terminal MD does not need to be zero because the warhead of the missile is detonated within R_D by the function of a proximity fuse, which is used in most tactical missiles. With the designed RHG the missile intercepts the target irrespective of t_{go} only if the condition $t_D < t_f$ is satisfied, whereas with the OGL the missile intercepts the target if t_{go} is accurate. A smaller t_D enables the missile to intercept the target in the presence of a larger uncertainty of t_f. However, too small t_D would cause saturation of the missile acceleration for a short time from the initial time if the initial lateral MD rate β is large. The effect of the initial lateral MD rate β decreases exponentially, as shown in Eq. (B1) of Appendix B as t increases. Thus, the effect of the target maneuvering dominates the lateral MD as t increases.

The acceleration of the proposed RHG is bounded as follows:

$$|A_{mn}(t)| \leq \begin{cases} \max\left\{\dfrac{N\beta}{T}, f_1(t_a^*), f_1(t_a^*) + f_2(t_b^*) - \beta K(t_b^*)\right\} & \text{for} \quad N > 4 \\ \max\left\{\dfrac{4\beta}{T}, (1 + 2e^{-2})\alpha\right\} & \text{for} \quad N = 4 \\ \max\left\{\dfrac{2N\beta}{T\sqrt{N(4-N)}}, \dfrac{4\alpha}{\sqrt{N(4-N)}}\right\} & \text{for} \quad N < 4 \end{cases} \tag{9}$$

where $K(t)$, $f_1(t)$, $f_2(t)$, t_a^*, and t_b^* are defined in Eqs. (B2), (B4), and (B5) of Appendix B, which gives the derivation of Eq. (9). As shown in Eqs. (8) and (9), a large N gives a small t_D but a large acceleration, while a large T gives a small acceleration but a large t_D. Although we can always find the pair (N, T) that satisfies Eqs. (8) and (9) for any t_D and R_D, we consider the acceleration bound in the selection of t_D, N, and T for real application.

There are many sets of t_D, N, and T that satisfy Eqs. (8) and (9). In the following, we propose one of the ways to select t_D, N, and T suitably at the initial time $t = 0$:

1) With measured $r(0)$ and $\dot{r}(0)$ at $t = 0$, we obtain the final time $t_f = -r(0)/\dot{r}(0)$. Because the target maneuvers arbitrarily and measured $r(0)$ and $\dot{r}(0)$ are not accurate, the final time t_f is not accurate. From the inaccurate t_f, we select t_D satisfying the condition $t_D < t_f^* < t_f$ where t_f^* is a variable for an efficient iteration.

2) For the given R_D and the selected t_D, we can easily find a set of the pair (N, T) that satisfies Eq. (8) in Theorem 1.

3) For the selected set of the pair (N, T), if Eq. (9) is not satisfied, we repeat the procedure 2) and 3).

4) If we cannot find a set of the pair (N, T) from the procedure 2) and 3), we change t_D to $t_D = t_D + (t_f^* - t_D)/2$ and repeat the procedure 2)–4).

5) If we fail in the procedure 4), we change t_f^* to $t_f^* = t_f^* + (t_f - t_f^*)/2$ and repeat the procedure 2)–5).

6) If we want to render the lateral MD below R_D within a faster time, we change t_f^* and t_D to $t_f^* = t_D$ and $t_D = t_D/2$, respectively, and repeat the procedure 2)–6).

We can avoid the infinite iteration by introducing finite-loop counters in each repetition of the procedure 3)–6).

III. Computer Simulation

Through three simulation examples the ability of the RHG to intercept the target is illustrated, and the performance of the RHG is compared with the OGL. The data in Table 1 are used in each of the simulations, and the assumption is made that the missile can intercept the target if the MD is less than 4 ft, measurements of $y(t)$ and $\dot{y}(t)$ have no error, the future target acceleration is not known, the initial time $t = 0$, the lateral MD $y(0) = 0$, and the missile can maneuver in the range of $|A_m(t)| \leq 30\ g$ where g is the acceleration caused by gravity.

Table 1 Parameters for simulation

Parameter	Value
V_m	Mach 2
V_t	Mach 1
$\gamma_t(0)$	180 deg
$r(0)$	5000 ft

A. Performance of RHG

In this simulation example Theorem 1 is illustrated. Assume that the target is maneuvering as follows:

$$A_t(t) = \begin{cases} 10\ g, & 0 \leq t < 1 \\ -10\ g, & t \geq 1 \end{cases}$$

$\gamma_m(0) = -1$ deg, and the missile is not on a collision course, i.e., $\dot{y}(0) \neq 0$. From the assumption

$$|A_{tn}(t)| \leq 10\ g (= \alpha)$$

$$|\dot{y}(0)| = |V_t \sin\ \gamma_t(0) - V_m \sin\ \gamma_m(0)| \leq 38.9 \text{ ft/s} (= \beta)$$

Then, the RHG is designed to keep the lateral MD less than 4 ft $(= R_D)$ from t_D to the intercept time by using the design procedure of the preceding section for t_D, T, and N.

First, we obtain $t_f = -[r(0)/\dot{r}(0)] \approx 1.5$ s, where $r(0)$ and $\dot{r}(0)$ are given in Table 1 and Eq. (1), respectively. We set t_D and t_f^* to 1 and 1.2 s, respectively. Then, for the given R_D and the selected t_D, the pair ($N = 4$, $T = 0.2$ s) satisfies Eqs. (8) and (9) because

$$\bar{y} = 3.217 \text{ ft} < 4 \text{ ft} \qquad \bar{A}_{mn} = 24.198\ g < 30\ g$$

where $\bar{y}$ is defined as $\bar{y} = t_D e^{-(2/T)t_D}\beta + \frac{T^2}{4}\alpha$ and $\bar{A}_{mn}$ is as $\bar{A}_{mn} = 4\beta/T$.

We repeat the design procedure 2)–6) for the selection of a smaller t_D. Let t_f^* and t_D be 1.0 and 0.5 s, respectively. Then, the pair ($N = 4$, $T = 0.2$ s) also satisfies Eqs. (8) and (9) because

$$\bar{y} = 3.346 \text{ ft} \qquad \bar{A}_{mn} = 24.198\ g$$

We repeat the design procedure 2)–6) again. Let t_f^* and t_D be 0.5 and 0.25 s, respectively. Then, Eq. (8) is not satisfied because $\bar{y} = 4.531$ ft. If we change T as $T = 0.15$ s, Eq. (8) is satisfied because $\bar{y} = 2.155$ ft. However, Eq. (9) is not satisfied because $\bar{A}_{mn} = 32.263\ g$. Here, we stop the design procedure because t_D is small enough to allow a large uncertainty of t_f although we may find a smaller t_D than 0.5 s from the repeated trial.

From Theorem 1 we know that $|y(t)| \leq 4$ ft for $t \geq 0.5$ s with the pair ($N = 4$, $T = 0.2$ s). Figure 2 shows that our purpose is satisfied. After about 1.5 s the MD r becomes smaller than 4 ft, i.e., the missile intercepts the target. Values of N larger than 4, or values of T smaller than 0.2 s, can render the lateral MD less than 4 ft.

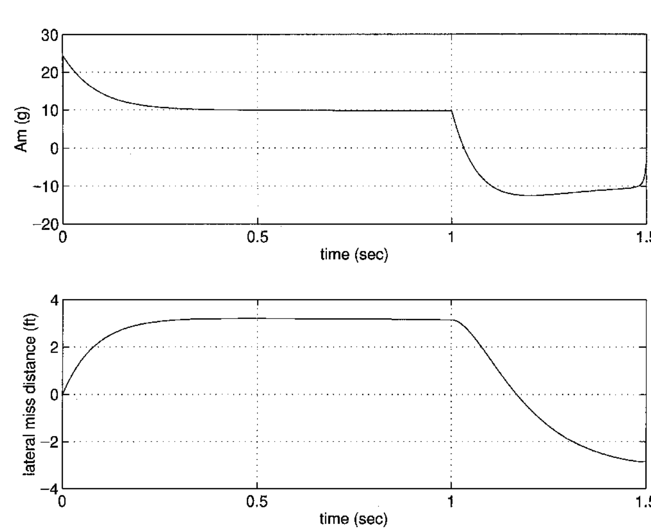

Fig. 2 Miss normal distance.

However, extremely large values of N and extremely small values of T might cause saturation of the RHG and thus a failure to intercept the target. In Fig. 2 the lateral MD $y(t)$ can be negative because it is a relative distance between the target and the missile.

B. Comparison of RHG and OGL for Inaccurate Time-to-Go

In this simulation example the performance of the RHG is compared with that of the OGL when t_{go} is inaccurate. Assume that $\gamma_m(0) = 0$ deg, $A_t(t) = 10\ g$, $N = 3$, $T = 0.2$ s, the missile is on a collision course at initial time, i.e., $\dot{y}(0) = 0$, and at each time t_{go} is computed by

$$t_{go} = -r(t)/\dot{r}(t)$$

where $t_{go} \approx 1.5$ s at initial time.

The t_{go} with estimation errors are modeled as follows:

$$\hat{t}_{go} = a t_{go} + b$$

where a is a scale factor error and b is a bias error. This error model can be found in Ref. 1. In the model the t_{go} is accurate when $a = 1$ and $b = 0$.

The terminal MD at $t = 1.5$ s is shown when $a = 0.7 + 0.1 * (k - 1)$ in Fig. 3 and $b = -0.15 + 0.05 * (k - 1)$ in Fig. 4 for $k = 1, 2, \ldots, 7$. t_{go} is accurate when $k = 4$. The simulation results for seven types of model errors are shown in Fig. 3 for each a when $b = 0$ and in Fig. 4 for each b when $a = 1$, respectively. As shown in Figs. 3 and 4, as errors in t_{go} increase the performance of the OGL degrades rapidly and the missile with the OGL fails to intercept the target, whereas the performance of the RHG is not affected by t_{go}. The failure to intercept the target occurs because the OGL is saturated when the time-to-go has errors. The saturated feedback of the OGL increases the terminal MD and then causes a failure to intercept the target, even if the time-to-go is accurate near the final time t_f.

C. Comparison of RHG and OGL for Accurate Time-to-Go

In this simulation example properties of the RHG and OGL are compared when the accurate t_{go} is given for the OGL. Parameters for simulation are the same as those for the second simulation example. Simulation results are shown in Figs. 5 and 6. Both RHG and OGL intercept the target near 1.5 s, which means that the RHG is not conservative compared with the OGL in terms of the intercept time. As shown in Fig. 6, the RHG increases rapidly from initial time until T horizon and decreases slowly after T horizon, keeping the lateral MD below R_D for $t \geq t_D$. The OGL becomes larger and larger as the missile approaches the target, making the terminal MD zero. The OGL ignores the lateral MD before t_f, even though it minimizes the cost function (3) from t to t_f, making the terminal MD zero.

The difference in performance between the RHG and the OGL comes from the following. The RHG tries to render the lateral MD zero after T horizon from the present time t. This trial is repeated as the time t goes on. It causes a rapid change in $y(t)$, i.e., large $\dot{y}(t)$ and large missile acceleration, from initial time until T horizon. Because $\dot{y}(t)$ is decreasing and $y(t)$ almost holds after t_D, the RHG decreases slowly. The OGL is inversely proportional to the change in the time-to-go. As the missile gets closer to the target, the time-to-go decreases and making the terminal MD zero requires rapid change of $y(t)$ near t_f. This causes the OGL to increase to a large value near t_f. Thus, if the time-to-go is inaccurate, there is a higher probability that the OGL will be saturated as the missile gets closer to the target. As just noted, there has been some research on

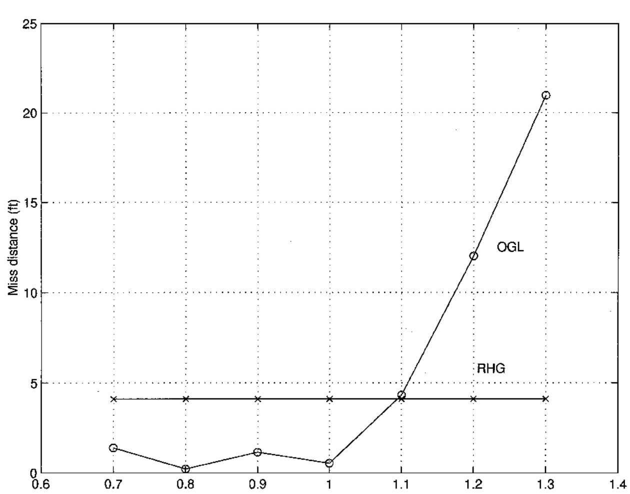

Fig. 3 Effect of a scale factor error in t_{go}.

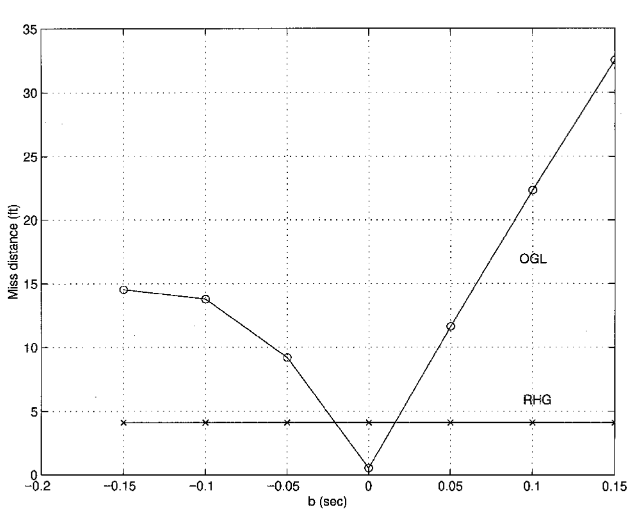

Fig. 4 Effect of a bias error in t_{go}.

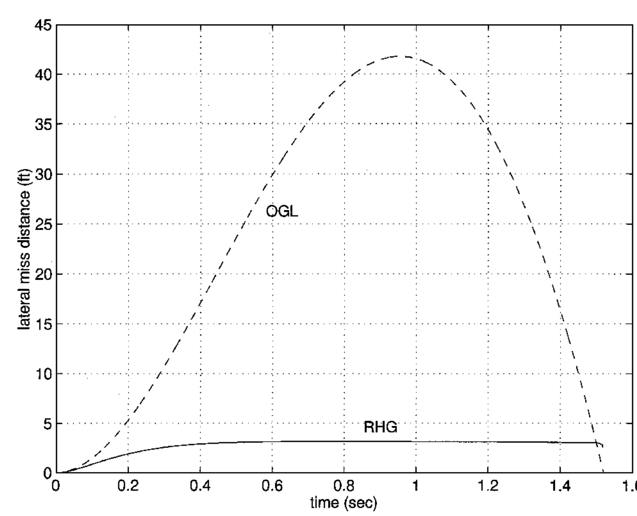

Fig. 5 Miss normal distance of RHG and OGL.

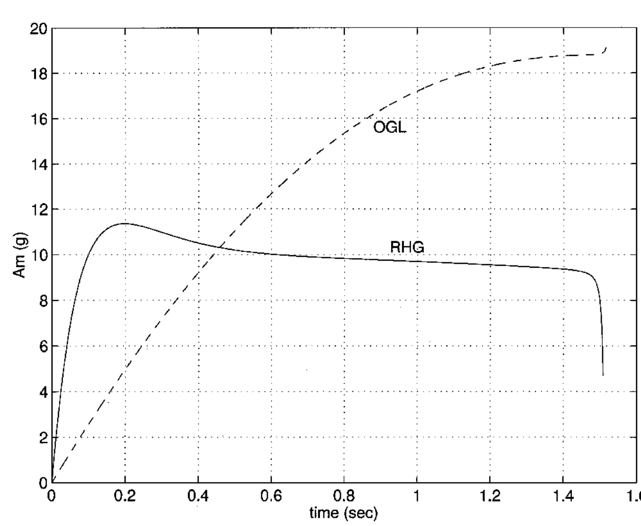

Fig. 6 Guidance law of RHG and OGL.

the accurate estimation of the time-to-go using additional measurement information and somewhat complicated algorithms. However, additional measurement information can also be contaminated by noise, and the time-to-go may not be estimated accurately in the available sampling time. For some short-term missions, especially for defensive weapons with limited capability for offense, tactical missiles have only a short operation time within which the time-to-go may not be estimated accurately because of lack of computation time.

IV. Conclusions

In this paper a new guidance law based on the receding horizon strategy is proposed that does not use the time-to-go. In the presence of arbitrary target maneuvers and an initial lateral MD rate between the target and the missile, the proposed RHG is shown to keep the lateral MD from the appropriately selected time to the intercept time less than the given value, within which the warhead of the missile is detonated without the time-to-go, whereas the OGL renders the terminal MD zero at the final time using the accurate time-to-go. The proposed RHG intercepts the target by adjusting the navigation constant and the cost horizon size. Through simulation, one can also see that the performance of the RHG is better than the OGL in terms of the terminal MD when the time-to-go is inaccurate and that the RHG is not conservative compared with the OGL in terms of the intercept time even when the time-to-go is accurate.

The performance of the OGL is dependent on the estimation of the time-to-go, and thus the OGL has a computational burden for estimating the time-to-go. For this reason the proposed RHG can be an appropriate guidance law when the time-to-go is inaccurate or unavailable, such as when the time-to-go cannot be estimated accurately in the available sampling time. The concept of the RHG may be applied not only to other guidance problems such as strategic missiles, missile midcourse guidance system, or underwater vehicles, but also to guidance applications such as path planning of robots, terrain avoidance of aircraft, or docking of spacecraft.

Appendix A: Proof of Theorem 1

To prove Theorem 1, we change Eq. (2) with Eq. (6) into

$$\dot{X} = \mathcal{A}X + GA_{\text{tn}} \tag{A1}$$

where

$$X = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} 0 & 1 \\ -N/T^2 & -N/T \end{bmatrix}$$

Eigenvalues λ_1 and λ_2 of $\mathcal{A}$ are

$$\lambda_1 = -\frac{N}{2T} + \frac{\sqrt{N(N-4)}}{2T}, \qquad \lambda_2 = -\frac{N}{2T} - \frac{\sqrt{N(N-4)}}{2T} \qquad \text{for} \quad N > 4$$

$$\lambda_1 = \lambda_2 = -\frac{2}{T} \qquad \text{for} \quad N = 4$$

$$\lambda_1 = -\frac{N}{2T} + \frac{\sqrt{N(4-N)}}{2T}i, \qquad \lambda_2 = -\frac{N}{2T} - \frac{\sqrt{N(4-N)}}{2T}i \qquad \text{for} \quad N < 4 \tag{A2}$$

We define the state transition matrix of Eq. (A1) as Φ, then $X(t)$ is

$$X(t) = \Phi(t)X(0) + \int_0^t \Phi(t-\tau)GA_{\text{tn}}(\tau)\,\mathrm{d}\tau \tag{A3}$$

where

$$\Phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix}$$

The elements of Φ for each N are given as follows.

1) Case of $N > 4$:

$$\phi_{11}(t) = \frac{e^{\lambda_1 t}\left(N + \sqrt{N(N-4)}\right) + e^{\lambda_2 t}\left(-N + \sqrt{N(N-4)}\right)}{2\sqrt{N(N-4)}}$$

$$\phi_{12}(t) = \frac{T\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right)}{\sqrt{N(N-4)}}, \qquad \phi_{21}(t) = \frac{N\left(-e^{\lambda_1 t} + e^{\lambda_2 t}\right)}{T\sqrt{N(N-4)}}$$

$$\phi_{22}(t) = \frac{e^{\lambda_1 t}\left(-N + \sqrt{N(N-4)}\right) + e^{\lambda_2 t}\left(N + \sqrt{N(N-4)}\right)}{2\sqrt{N(N-4)}}$$

2) Case of $N = 4$:

$$\phi_{11}(t) = [1 + (2/T)t]e^{-(2/T)t}, \qquad \phi_{12}(t) = te^{-(2/T)t}$$

$$\phi_{21}(t) = -(4/T^2)te^{-(2/T)t}, \qquad \phi_{22}(t) = [1 - (2/T)t]e^{-(2/T)t}$$

3) Case of $N < 4$:

$$\phi_{11}(t) = e^{-(N/2T)t}\left[\cos\frac{\sqrt{N(4-N)}}{2T}t + \frac{N}{\sqrt{N(4-N)}}\sin\frac{\sqrt{N(4-N)}}{2T}t\right]$$

$$\phi_{12}(t) = e^{-(N/2T)t}\left[\frac{2T}{\sqrt{N(4-N)}}\sin\frac{\sqrt{N(4-N)}}{2T}t\right]$$

$$\phi_{21}(t) = -(N/T^2)\phi_{12}(t)$$

$$\phi_{22}(t) = e^{-(N/2T)t}\left[\cos\frac{\sqrt{N(4-N)}}{2T}t - \frac{N}{\sqrt{N(4-N)}}\sin\frac{\sqrt{N(4-N)}}{2T}t\right]$$

The values $\dot{y}(0)$ and $A_{\text{tn}}(t)$ affect $y(t)$ and $A_{\text{mn}}(t)$ independently.

First, we consider the effect of $\dot{y}(0)$ under conditions such as $y(0) = 0$, $A_{\text{tn}}(t) = 0$ for $\forall t$, and $|\dot{y}(0)| \le \beta$. From Eq. (A3), because $y(t) = \phi_{12}(t)\dot{y}(0)$,

$$|y(t)| \le \begin{cases} \dfrac{T}{\sqrt{N(N-4)}}\exp\left[\dfrac{-N + \sqrt{N(N-4)}}{2T}t\right]\beta & \\ \quad \text{because} \quad e^{\lambda_1 t} - e^{\lambda_2 t} \ge 0 & \text{for} \quad N > 4 \\ te^{-(2/T)t}\beta & \text{for} \quad N = 4 \\ \dfrac{2T}{\sqrt{N(4-N)}}e^{-(N/2T)t}\beta & \text{for} \quad N < 4 \end{cases}$$

Second, we consider the effect of $A_{\text{tn}}(t)$ under conditions such as $y(0) = 0$, $\dot{y}(0) = 0$, and $|A_{\text{tn}}(t)| \le \alpha$. From Eq. (A3), because

$$y(t) = \int_0^t \phi_{12}(t-\tau)A_{\text{tn}}(\tau)\,\mathrm{d}\tau$$

$$|y(t)| \le \alpha\int_0^t |\phi_{12}(t-\tau)|\,\mathrm{d}\tau \tag{A4}$$

From Eq. (A2) because λ_1, $\lambda_2 < 0$ and $|\lambda_1| < |\lambda_2|$, $\phi_{12}(t) \ge 0$, thus from Eq. (A4),

$$|y(t)| \le \begin{cases} \dfrac{T^2}{N}\alpha \quad \text{because} \quad 0 \le \phi_{11}(t) \le 1 & \text{for} \quad N > 4 \\ \dfrac{T^2}{4}\alpha & \text{for} \quad N = 4 \\ \dfrac{4T^2}{N\sqrt{N(4-N)}}\alpha & \text{for} \quad N < 4 \end{cases}$$

Therefore, when $y(0) = 0$, $|\dot{y}(0)| \le \beta$, and $|A_{\text{tn}}(t)| \le \alpha$, the MD is bounded as

$$|y(t)| \le \begin{cases} \left\{\dfrac{T}{\sqrt{N(N-4)}}\exp\left[\dfrac{-N+\sqrt{N(N-4)}}{2T}t\right]\beta + \dfrac{T^2}{N}\alpha\right\} & \text{for} \quad N > 4 \\ \left[te^{-(2/T)t}\beta + \dfrac{T^2}{4}\alpha\right] & \text{for} \quad N = 4 \\ \left[\dfrac{2T}{\sqrt{N(4-N)}}e^{-(N/2T)t}\beta + \dfrac{4T^2}{N\sqrt{N(4-N)}}\alpha\right] & \text{for} \quad N < 4 \end{cases} \tag{A5}$$

In Eq. (A5) the values of functions for $N > 4$ and $N < 4$ are decreasing as t increases. The function for $N = 4$ has a maximum value when $t = T/2$.

Appendix B: Derivation of Eq. (9)

From Eqs. (6) and (A3)

$$|A_{\mathrm{mn}}(t)| \le A^1_{\mathrm{mn}} + A^2_{\mathrm{mn}}$$

where

$$A^1_{\mathrm{mn}} = \left(\frac{N}{T^2}\right)|\phi_{12}(t)\dot{y}(0) + T\phi_{22}(t)\dot{y}(0)|$$

$$A^2_{\mathrm{mn}} = \frac{N}{T^2}\left|\int_0^t \phi_{12}(t-\tau)A_{\mathrm{tn}}(\tau) + T\phi_{22}(t)A_{\mathrm{tn}}(\tau)\,\mathrm{d}\tau\right|$$

$$t_b^* = \frac{T}{\sqrt{N(N-4)}}\ln\left\{\frac{\left[N+\sqrt{N(N-4)}\right]^2(\alpha T - N\beta) + 2N\beta\left[N+\sqrt{N(N-4)}\right]}{\left[N-\sqrt{N(N-4)}\right]^2(\alpha T - N\beta) + 2N\beta\left[N-\sqrt{N(N-4)}\right]}\right\} \tag{B5}$$

First, we consider the effect of $\dot{y}(0)$ under conditions such as $y(0) = 0$, $A_{\mathrm{tn}}(t) = 0$ for $\forall t$ and $|\dot{y}(0)| \le \beta$. From Eq. (A3)

$$A^1_{\mathrm{mn}} \le \begin{cases} \beta|K(t)| & \text{for} \quad N > 4 \\ \dfrac{4\beta}{T^2}\left|(T-t)e^{-(2/T)t}\right| & \text{for} \quad N = 4 \\ \dfrac{2N\beta}{T\sqrt{N(4-N)}}e^{-(N/2T)t} & \text{for} \quad N < 4 \end{cases} \tag{B1}$$

where

$$K(t) = \left[N/2T\sqrt{N(N-4)}\right][(2-N)(e^{\lambda_1 t} - e^{\lambda_2 t}) + \sqrt{N(N-4)}(e^{\lambda_1 t} + e^{\lambda_2 t})] \tag{B2}$$

Second, we consider the effect of $A_{\mathrm{tn}}(t)$ under conditions such as $y(0) = 0$, $\dot{y}(0) = 0$, and $|A_{\mathrm{tn}}(t)| \le \alpha$. From Eq. (A3)

$$A^2_{\mathrm{mn}} \le \begin{cases} f_1(t_a^*) + f_2(t) & \text{for} \quad N > 4 \quad \text{and} \quad t \ge t_a^* \\ f_3(t) & \text{for} \quad N > 4 \quad \text{and} \quad 0 \le t < t_a^* \\ \alpha\left[1 + 2e^{-2} + \left(1 - \dfrac{2}{T}t\right)e^{-(2/T)t}\right] & \text{for} \quad N = 4 \quad \text{and} \quad t \ge T \\ \alpha\left[1 + \dfrac{1}{T}(2t - T)e^{-(2/T)t}\right] & \text{for} \quad N = 4 \quad \text{and} \quad 0 \le t < T \\ \dfrac{4\alpha}{\sqrt{N(4-N)}}\left[1 - e^{-(N/2T)t}\right] & \text{for} \quad N < 4 \end{cases} \tag{B3}$$

where

$$f_1(t) = \alpha\left[1 - (e^{\lambda_1 t} + e^{\lambda_2 t}) + \frac{N}{\sqrt{N(N-4)}}(e^{\lambda_1 t} - e^{\lambda_2 t})\right]$$

$$f_2(t) = \frac{\alpha}{2} - \frac{1}{2}f_1(t), \qquad f_3(t) = \frac{\alpha}{2} + \frac{1}{2}f_1(t)$$

$$t_a^* = \frac{T}{\sqrt{N(N-4)}}\ln\left[\frac{N - 2 + \sqrt{N(N-4)}}{N - 2 - \sqrt{N(N-4)}}\right] \tag{B4}$$

Here, t_a^* satisfies $K(t_a^*) = 0$, and $\ln$ implies that when $e^a = b$, $a = \ln b$.

Now, from the preceding equations the upper bound of $|A_{\mathrm{mn}}(t)|$ for each N are given as follows:

1) Case of $N > 4$:

When $t \ge t_a^*$, $|A_{\mathrm{mn}}(t)| \le -\beta K(t) + f_1(t_a^*) + f_2(t)$. Note that $f_2(t) < 0$ for $t \ge t_a^*$ while $-\beta K(t) \ge 0$ and $f_1(t_a^*) \ge 0$. Also note that t_b^* satisfies $\dot{f}_2(t_b^*) - \beta\dot{K}(t_b^*) = 0$ where

When $0 \le t < t_a^*$, $|A_{\mathrm{mn}}(t)| \le \beta K(t) + f_3(t)$. Note that $f_3(t)$ is a monotonic increasing function because $\dot{f}_3(t) > 0$ for $0 \le t < t_a^*$.

From the preceding equations the upper bound of $|A_{\mathrm{mn}}(t)|$ for $N > 4$ is given as

$$|A_{\mathrm{mn}}(t)| \le \begin{cases} f_1(t_a^*) & \text{for} \quad \alpha \ge \alpha_1 \\ \max\left\{N\beta/T,\ f_1(t_a^*)\right\} & \text{for} \quad \alpha_2 \le \alpha < \alpha_1 \\ \max\left\{N\beta/T,\ f_1(t_a^*) + f_2(t_b^*) - \beta K(t_b^*)\right\} & \text{for} \quad \alpha < \alpha_2 \end{cases}$$

where

$$\alpha_1 = \frac{N\beta\left[N - 2 + \sqrt{N(N-4)}\right]}{T\left[N + \sqrt{N(N-4)}\right]}, \qquad \alpha_2 = \frac{N\beta\left[N - 2 - \sqrt{N(N-4)}\right]}{T\left[N - \sqrt{N(N-4)}\right]}$$

2) Case of $N = 4$:

$$|A_{\mathrm{mn}}(t)| \leq \begin{cases} [1 + 2e^{-2}]\alpha & \text{for} \quad \alpha \geq 3\beta/T \\ 4\beta/T & \text{for} \quad \alpha < 3\beta/T \end{cases}$$

3) Case of $N < 4$:

$$|A_{\mathrm{mn}}(t)| \leq \begin{cases} \dfrac{4\alpha}{\sqrt{N(4-N)}} & \text{for} \quad \alpha \geq \dfrac{N\beta}{2T} \\ \dfrac{2N\beta}{T\sqrt{N(4-N)}} & \text{for} \quad \alpha < \dfrac{N\beta}{2T} \end{cases}$$

From 1)–3), when $y(0) = 0$, $|\dot{y}(0)| \leq \beta$, and $|A_{\mathrm{tn}}(t)| \leq \alpha$, Eq. (9) is satisfied.

References

[1]Nesline, F. W., and Zarchan, P., "A New Look at Classical vs Modern Homing Missile Guidance," *Journal of Guidance and Control*, Vol. 4, No. 1, 1979, pp. 78–85.

[2]Ha, I. J., Hur, J. S., Ko, M. S., and Song, T. L., "Performance Analysis of PNG Laws for Randomly Maneuvering Targets," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 26, No. 5, 1990, pp. 713–721.

[3]Holder, E. J., and Sylveser, V. B., "An Analysis of Modern Versus Classical Homing Guidance," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 26, No. 4, 1990, pp. 599–606.

[4]Garber, V., "Optimum Intercept Laws for Accelerating Targets," *AIAA Journal*, Vol. 6, No. 11, 1968, pp. 2196–2198.

[5]Cottrell, R. G., "Optimal Intercept Guidance for Short-Range Tactical Missiles," *AIAA Journal*, Vol. 9, No. 7, 1971, pp. 1414, 1415.

[6]Asher, R. B., and Matuszewski, J. P., "Optimal Guidance with Maneuvering Targets," *Journal of Spacecraft and Rockets*, Vol. 11, No. 3, 1974, pp. 204–206.

[7]Nazaroff, G. J., "An Optimal Terminal Guidance Law," *IEEE Transaction on Automatic Control*, Vol. 21, No. 6, 1976, pp. 407, 408.

[8]Rusnak, I., and Meir, L., "Optimal Guidance for High-Order and Acceleration Constrained Missile," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 3, 1991, pp. 589–596.

[9]Cho, H., and Ryoo, C. K., "Closed-Form Optimal Guidance Law for Missiles of Time-Varying Velocity," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 5, 1996, pp. 1017–1022.

[10]Lee, G. K. F., "Estimation of the Time-to-Go Parameter for Air-to-Air Missiles," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 2, 1985, pp. 262–266.

[11]Lin, C. F., *Modern Navigation, Guidance, and Control Processing*, Prentice–Hall, Upper Saddle River, NJ, 1991, pp. 35–37.

[12]Hull, D. G., Radke, J. J., and Mack, R. E., "Time-to-Go Prediction for Homing Missiles Based on Minimum-Time Intercepts," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 865–871.

[13]Kwon, W. H., and Pearson, A. E., "A Modified Quadratic Cost Problem and Feedback Stabilization of a Linear System," *IEEE Transaction on Automatic Control*, Vol. 22, No. 5, 1977, pp. 838–842.

[14]Kim, K. B., Lee, J. W., and Kwon, W. H., "Intervalwise Receding Horizon H_∞ Tracking Control for Discrete Linear Periodic Systems," *IEEE Transaction on Automatic Control* (to be published).

[15]Garcia, C. E., Prett, D. M., and Morari, M., "Model Predictive Control: Theory and Practive—A Survey," *Automatica*, Vol. 25, No. 3, 1989, pp. 335–348.